# Eigenvalue problems and special functions

*Dissertation submitted to*
*Visvesvaraya National Institute of Technology, Nagpur*
*in partial fulfillment of the requirement for the award of*
*the degree*

## Master of Science
## in
## Mathematics

*by*

**SATYANAM RAJPOOT**
**(MS23MTH017)**

under the guidance of

**Dr. MOHAN KUMAR MALLICK**

**Department of Mathematics**
**Visvesvaraya National Institute of Technology**
**Nagpur 440010 (India)**

**May, 2025**

# Eigenvalue problems and special functions

***Dissertation submitted to***
***Visvesvaraya National Institute of Technology, Nagpur***
***in partial fulfillment of the requirement for the award of***
***the degree***

## Master of Science
## in
## Mathematics

***by***

**SATYANAM RAJPOOT**
**(MS23MTH017)**

under the guidance of

**Dr. MOHAN KUMAR MALLICK**

**Department of Mathematics**
**Visvesvaraya National Institute of Technology**
**Nagpur 440010 (India)**

**2025**

©Visvesvaraya National Institute of Technology (VNIT) 2025

**Department of Mathematics**
**Visvesvaraya National Institute of Technology, Nagpur**

# Declaration

I, Satyanam Rajpoot, hereby declare that this dissertation titled **"Eigenvalue problems and special functions "** is carried out by me in the Department of Mathematics of Visvesvaraya National Institute of Technology, Nagpur. The work is original and has not been submitted earlier whole or in part for the award of any degree/diploma at this or any other Institution/University.

**Satyanam Rajpoot**

Date: 31/05/2025

# Certificate

This is to certify that the dissertation titled **"Eigenvalue problems and special functions "** is submitted by **Satyanam Rajpoot** in partial fulfillment of the requirements for the award of the degree of **Master of Science in Mathematics**, VNIT Nagpur. The work is comprehensive, complete and fit for final evaluation.

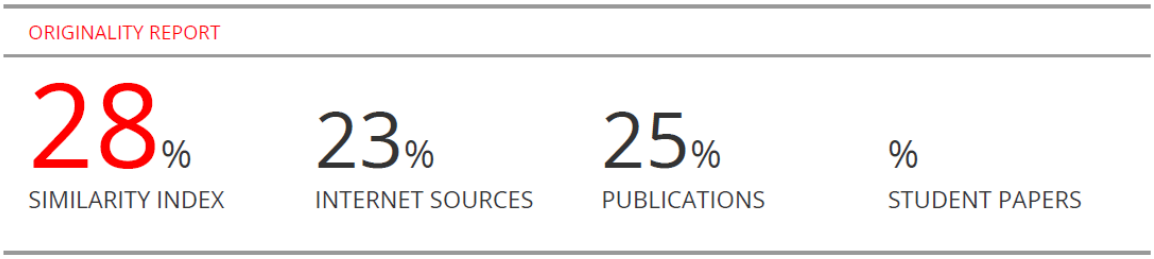
**Dr. Mohan kumar mallick**
Assistant Professor
Department of Mathematics
VNIT, Nagpur

Head, Department of Mathematics
VNIT, Nagpur
Date:31/05/2025

**VISVESVARAYA NATIONAL INSTITUTE OF TECHNOLOGY NAGPUR**

## Non-Plagiarism Certificate

Certified that M.Sc. Dissertation titled **"Eigenvalue problems and special functions "** submitted by **Satyanam rajpoot, Enrollment No. MS23MTH017, ID No. 30577**, is original in nature and not copied from any literature material. Due credit has been given to the original researcher wherever referred.




Signature:
Student Name: Satyanam Rajpoot
Enrollment No: MS23MTH017
ID No: 30577

**Dr. Mohan Kumar Mallick**
Assistant Professor
Department of Mathematics
VNIT, Nagpur

# Acknowledgements

It is of great pleasure and privilege to express my deep sense of gratitude to my supervisor **Dr Mohan Kumar Mallick**, Assistant Professor, Department of Mathematics, Visvesvaraya National Institute of Technology, Nagpur, for his encouraging guidance and providing mentorship throughout my project work. This work would not have come into existence in the absence of his valuable guidance. His guidance throughout this project work right from assigning project topic, guiding at every stage of my project, giving valuable advice and all the useful suggestion and brainstorming sessions.

I convey my deep sense of gratitude to **Prof Pallavi Mahale**, Head, Department of Mathematics, VNIT, Nagpur, and my project evaluation committee **Prof. P.P Chakaravarthy** and **Dr. Purnima Satapathy** . I take this moment to shower my words of gratitude on all the faculty members of the Department of Mathematics, who have supported me whenever I was in need of their help. I would like to thank **Mr Sachin Singh** and **Mr Karan Rathore** Ph.D Research Scholars of the Mathematics Department, for their valuable time, suggestions, and discussions when needed.
I would like to thank my family for always keeping me in their prayers and my parents **Rampal rajpoot** and **Sumitra rajpoot**, for always being enough for me throughout my life. I would also like to thank my friends Anup Das, Abhishek Dubey, Rohan Naitame, Nitesh Prajapati, Mehboob Alam and Raju Kumar **M.Sc Students** for their continuous support. A big thanks to my sister **Meena rajpoot** for the mental and emotional support she rendered throughout the tough times.

I sincerely thank VNIT for their support to conduct my project work at the Department of Mathematics.

Finally, I bow down before the almighty who has made everything possible.

**Satyanam Rajpoot**

# Abstract

This dissertation studied the mathematical structure and implications of eigenvalue problems, with a specific focus on Sturm Liouville(SL) systems and BVPs. The analysis begins by reformulating second-order linear differential equations through the method of separation of variables, leading to the SL form. The theoretical framework is developed by introducing self-adjoint operators and establishing conditions under which eigenvalues and eigenfunctions exist and possess orthogonality properties with respect to appropriate weight functions.

Classical examples are explored to illustrate the distinctions between regular and singular SL problems. Particular emphasis is placed on how singularities arise due to infinite intervals or vanishing coefficients, as seen in Legendre-type equations. Various boundary conditions—including Dirichlet, Neumann, Robin, and mixed types—are discussed in detail, alongside their impact on solution behavior and uniqueness.

Additionally, analytical techniques are applied to solve BVPs in geometrically significant domains such as circles, annuli, and rectangles. These problems are addressed using Fourier series and separation of variables, providing explicit solutions under given boundary conditions. The dissertation further employs maximum and minimum principles demonstrate the stability and uniqueness of two-dimensional Laplace equation solutions.

Overall, this work not only develops a rigorous understanding of the theoretical aspects of eigenvalue problems and boundary conditions but also illustrates their practical applications in mathematical physics and engineering.

# Contents


**1 Introduction** **1**

1.1 Background and Context . . . . . 1

1.2 Importance of SLTheory . . . . . 1

1.3 Role of BVPs . . . . . 1

1.4 Applications in Mathematical Physics . . . . . 2

1.5 Objectives of the Study . . . . . 2

**2 Eigenvalue Problems and Special functions** **3**

2.1 Sturm–Liouville Systems . . . . . 3

2.2 Observation of Eigenvalues and Eigenfunctions . . . . . 10

2.3 Some important Theorems . . . . . 10

2.4 Singular Sturm–Liouville Systems . . . . . 12

2.5 Example Related to Singular Sturm–Liouville Systems . . . . . 13

**3 BVPs and its Applications** **15**

3.0.1 $1^{st}$ BVP (The Dirichlet Problem) . . . . . 15

3.0.2 $2^{nd}$ BVP (The Neumann problem) . . . . . 15

3.0.3 $3^{rd}$ BVP (The Robin problem) . . . . . 16

3.0.4 $4^{th}$ BVP (Mixed problem) . . . . . 16

3.1 Maximum and Minimum Principles . . . . . 16

3.1.1 The Maximum Principle: . . . . . 16

3.1.2 Minimum Principle . . . . . 17

3.2 Dirichlet Problem in a Circular Domain (Interior Problem) . . . . . 19

3.3 Dirichlet Problem for a Circular Annulus . . . . . 21

3.4 Dirichlet Problem for a Rectangle . . . . . 23

3.5 Neumann Problem for a Circle . . . . . 25

**4 Conclusion and Future Scopes** **27**

4.1 Conclusion . . . . . 27

4.2 Future Scope . . . . . 27
**Bibliography** . . . . . .

# Abbreviations Used

| | |
|---|---|
| PDE | : PDE |
| ODE | : Ordinary Differential Equation |
| BC | : Boundary Condition |
| IVP | : Initial Value Problem |
| LHS | : Left-Hand Side |
| RHS | : Right-Hand Side |
| SL | : Sturm-Liouville |
| BVP | : BVPs |

# Chapter 1

# Introduction

## 1.1 Background and Context

Differential equations form the cornerstone of mathematical modeling in physics, engineering, and applied sciences. Among them, second-order linear differential equations with boundary conditions—known as BVPs, which play a critical role in describing a wide range of physical phenomena. Solutions to these problems often lead to eigenvalue problems, where the determination of characteristic values and associated functions becomes essential. In particular, SL theory provides a rigorous framework for studying such systems systematically.

## 1.2 Importance of SLTheory

When we solve PDEs by the method of seperation of variables, then Sturm-Liouville problems occurs. This theory not only yields eigenvalues and orthogonal eigenfunctions, but also allows us to expand arbitrary functions with regards to these eigenfunctions. These expansions are crucial in solving time-independent problems in mathematical physics. Infact SL theory constitutes a fundamental tool in the spectral analysis of linear operators and is central to Fourier-type expansions in applied mathematics.

## 1.3 Role of BVPs

BVPs represent mathematical formulations where solutions to differential equations must satisfy specific conditions at the domain boundaries. These include:

- Dirichlet problems — where the function values are prescribed on the boundary.

- Neumann problems — involving conditions on the normal derivative of the function.
- Robin (or mixed) problems — which combine Dirichlet and Neumann conditions.

Such formulations are key to modeling steady-state heat conduction, electrostatics, and fluid flow.

## 1.4 Applications in Mathematical Physics

Numerous classical equations—such as Laplace's, Helmholtz's, and the wave equations—require solving BVPs in specific geometries. For example, in circular or rectangular domains, the use of polar or cartesian coordinates respectively simplifies the process. The resulting ordinary differential equations often fall under the category of SL problems. This approach facilitates the exact solution of PDEs under well-posed boundary conditions, offering precise models for real-world systems.

## 1.5 Objectives of the Study

This dissertation aims to:

- Develop a solid understanding of SL theory and its properties.
- Analyze classical and singular SL systems.
- Solve BVPs in standard geometries such as circles, annuli, and rectangles.
- Explore the implications of maximum and minimum principles in proving uniqueness and continuity.

# Chapter 2

# Eigenvalue Problems and Special functions

Eigenvalues and eigenfunctions arise in various mathematical contexts, especially in linear algebra and differential equations. This chapter provides an overview of the motivation and significance of the topic.

## 2.1 Sturm–Liouville Systems

While solving PDEs using separation of variable method, we often deal with special types of problems called eigenvalue problems. A more general form of this method is known as SL theory.

Assuming separable variables, a second-order homogeneous PDE can be simplified to a ordinary differential equations [1]:

$$d_1 X'' + d_2 X' + (d_3 - \mu)X = 0, \tag{2.1}$$

$$e_1 Y'' + e_2 Y' + (e_3 + \mu)Y = 0. \tag{2.2}$$

Generally,

$$d_1(x)\frac{d^2\eta}{dx^2} + d_2(x)\frac{d\eta}{dx} + [d_3(x) + \mu]\,\eta = 0. \tag{2.3}$$

To transform this into Sturm–Liouville form, we introduce the following substi-

tutions:

$$\left.\begin{aligned} l(x) &= \exp\left(\int \frac{d_2(x)}{d_1(x)}\,dx\right), \\ m(x) &= \frac{d_3(x)}{d_1(x)l(x)}, \\ n(x) &= \frac{l(x)}{d_1(x)} \end{aligned}\right\} \tag{2.4}$$

Substituting into equation (2.3) yields the standard Sturm–Liouville form:

$$\frac{d}{dx}\left(l(x)\frac{d\eta}{dx}\right) + (m(x) + \mu n(x))\,\eta = 0. \tag{2.5}$$

This is known as the Sturm–Liouville differential equation.

In terms of the Sturm–Liouville operator, we define:

$$L \equiv \frac{d}{dx}\left(l(x)\frac{d}{dx}\right) + m(x).$$

Equation (2.5) can then be written concisely as:

$$L[\eta] + \mu n(x)\eta = 0. \tag{2.6}$$

Let $\mu$ be a constant that does not depend on the variable $x$. Let $l(x)$, $m(x)$, and $n(x)$ are real valued functions which is defined $[a, b] \subset \mathbb{R}$. To guarantee the existence of solutions to the associated problem, we make the following assumptions

- $m(x)$ and $n(x)$ are continuous functions in the interval $[a, b]$,
- $l(x)$ is continuously differentiable in the range $[a, b]$, i.e., $l(x) \in C^1([a, b])$.

A Sturm–Liouville differential equation is considered *regular* within $[a, b]$ if the following conditions are satisfied:

- $l(x)$ is strictly positive for all $x \in [a, b]$, i.e., $l(x) > 0$,
- The weight function $n(x)$ is also strictly positive on $[a, b]$, i.e., $n(x) > 0$.

Provided these regularity conditions hold, it follows that for any fixed value of the parameter $\mu$, the Sturm–Liouville equation admits two linearly independent solutions on the interval $[a, b]$.

The regular Sturm–Liouville problem consists of the differential equation

$$L[\eta] + \mu n(x)\eta = 0, \quad a \le x \le b,$$

along with the separated boundary conditions

$$d_1\eta(a) + d_2\eta'(a) = 0, \quad e_1\eta(b) + e_2\eta'(b) = 0,$$

where $d_1$ and $d_2$ (as well as $e_1$ and $e_2$) are real constants, not both zero in each pair.

In the framework of Sturm–Liouville theory, those specific values of the parameter $\mu$ that yield non-trivial solutions to the differential equation are called *eigenvalues*. The corresponding non-zero functions that satisfy the equation for these values of $\mu$ are referred to as eigen functions.

For a regular Sturm–Liouville problem, the domain of the operator $L$, denoted $D(L)$, is defined as:

$$D(L) = \left\{ \eta \in L^2([a,b]) \;\middle|\; \eta'' \in L^2([a,b]) \text{ and boundary conditions are satisfied} \right\}.$$

Here, $L^2([a,b])$ denotes the space of square-integrable functions over the interval $[a,b]$.

**Example:** Consider the regular Sturm–Liouville problem

$$\frac{d^2\eta}{dx^2} + \mu\eta = 0, \quad 0 \le x \le \pi,$$

subject to the boundary conditions:

$$\eta(0) = 0, \quad \eta'(\pi) = 0.$$

**Solution:** First, consider the case when $\mu \le 0$. In this situation, one can verify that the only solution satisfying both boundary conditions is the trivial solution $\eta(x) = 0$. Hence, such values of $\mu$ do not correspond to eigenvalues.

Now, assume $\mu > 0$, and define $\mu = \beta^2$ for some $\beta > 0$. The general solution of the differential equation

$$\eta'' + \beta^2\eta = 0$$

is

$$\eta(x) = C\cos(\beta x) + D\sin(\beta x),$$

where $D$ and $E$ are constants.

Applying the boundary condition $\eta(0) = 0$, we get:

$$\eta(0) = D\cos(0) + E\sin(0) = D = 0.$$

So, the solution reduces to

$$\eta(x) = E\sin(\beta x).$$

Next, we apply the second boundary condition $\eta'(\pi) = 0$. Differentiating $\eta(x)$, we find:

$$\eta'(x) = E\beta\cos(\beta x),$$

$$\eta'(\pi) = E\beta\cos(\beta\pi) = 0.$$

To ensure a nontrivial solution, we must have $E \neq 0$ and $\beta > 0$, which implies

$$\cos(\beta\pi) = 0.$$

This condition holds when:

$$\beta\pi = \left(k - \tfrac{1}{2}\right)\pi, \quad \text{for } k = 1, 2, 3, \ldots$$

Solving for $\beta$, we obtain:

$$\beta = \frac{2k-1}{2}, \quad k = 1, 2, 3, \ldots$$

and therefore,

$$\mu_k = \beta^2 = \left(\frac{2k-1}{2}\right)^2 = \frac{(2k-1)^2}{4}.$$

The corresponding eigenfunctions are given by:

$$\eta_k(x) = \sin\left(\frac{2k-1}{2}x\right), \quad k = 1, 2, 3, \ldots$$

**Example:** Let us analyze the second-order differential equation in Euler form given below

$$x^2\frac{d^2\eta}{dx^2} + x\frac{d\eta}{dx} + \mu\eta = 0, \quad \text{for } x \in [1, e].$$

This equation is accompanied by the boundary conditions

$$\eta(1) = 0, \quad \eta(e) = 0.$$

**Solution:** The given equation can be rewritten in Sturm–Liouville form. Dividing both sides by $x$, we get

$$\frac{d}{dx}\left(x\frac{d\eta}{dx}\right) + \mu x\eta = 0.$$

In this form, the functions are identified as

$$l(x) = x, \quad n(x) = x, \quad m(x) = 0.$$

Assuming $\mu > 0$, let us write $\mu = \beta^2$. The equation becomes

$$\frac{d}{dx}\left(x\frac{d\eta}{dx}\right) + \beta^2 x\eta = 0.$$

A common method for solving Euler-type equations is to try solutions of the form

$$\eta(x) = x^r.$$

For $\mu > 0$, the characteristic roots are imaginary. Therefore, the general solution is

$$\eta(x) = c_1 x^{i\beta} + c_2 x^{-i\beta}.$$

Using the relation

$$x^{i\beta} = e^{i\beta \ln x} = \cos(\beta \ln x) + i \sin(\beta \ln x),$$

we can write the solution in terms of real functions

$$\eta(x) = D \cos(\beta \ln x) + E \sin(\beta \ln x),$$

where $D$ and $E$ are real constants.

Applying the first boundary condition $\eta(1) = 0$, we find:

$$\eta(1) = D \cos(\beta \cdot 0) + E \sin(\beta \cdot 0) = D = 0,$$

since $\ln 1 = 0$. This simplifies the solution to

$$\eta(x) = E \sin(\beta \ln x).$$

Next, applying the second boundary condition $\eta(e) = 0$

$$\eta(e) = E \sin(\beta \ln e) = E \sin(\beta) = 0.$$

For a nonzero solution ($E \neq 0$), it follows that

$$\sin(\beta) = 0 \implies \beta = n\pi, \quad n = 1, 2, 3, \ldots$$

Thus, the eigenvalues are

$$\mu_n = \beta^2 = n^2 \pi^2, \quad n = 1, 2, 3, \ldots,$$

and the associated eigenfunctions take the form

$$\eta_n(x) = \sin(n\pi \ln x), \quad n = 1, 2, 3, \ldots$$

**Example:** Consider the second-order differential equation

$$\frac{d^2\eta}{dx^2} + \mu\eta = 0, \quad -\pi \leq x \leq \pi,$$

with the periodic boundary conditions

$$\eta(-\pi) = \eta(\pi), \qquad \eta'(-\pi) = \eta'(\pi).$$

**Solution:** The given problem fits the Sturm–Liouville framework with periodic boundary conditions, since the coefficient function $l(x) = 1$ satisfies $l(-\pi) = l(\pi)$.

## Case 1: $\mu > 0$

Let $\mu = \beta^2$, where $\beta > 0$. The general solution of the differential equation is

$$\eta(x) = A\cos(\beta x) + B\sin(\beta x),$$

where $A$ and $B$ are constants.

Applying the boundary condition $\eta(-\pi) = \eta(\pi)$, we compute

$$\eta(-\pi) = A\cos(\beta\pi) - B\sin(\beta\pi), \quad \eta(\pi) = A\cos(\beta\pi) + B\sin(\beta\pi).$$

Setting these equal gives

$$A\cos(\beta\pi) - B\sin(\beta\pi) = A\cos(\beta\pi) + B\sin(\beta\pi),$$

which simplifies to

$$2B\sin(\beta\pi) = 0.$$

Next, compute the derivatives

$$\eta'(-\pi) = -A\beta\sin(\beta\pi) + B\beta\cos(\beta\pi), \quad \eta'(\pi) = -A\beta\sin(\beta\pi) - B\beta\cos(\beta\pi).$$

The condition $\eta'(-\pi) = \eta'(\pi)$ gives

$$-A\beta\sin(\beta\pi) + B\beta\cos(\beta\pi) = -A\beta\sin(\beta\pi) - B\beta\cos(\beta\pi),$$

which leads to

$$2B\beta\cos(\beta\pi) = 0.$$

Together, these conditions yield the system

$$\sin(\beta\pi) = 0 \quad \text{and} \quad \cos(\beta\pi) = 0.$$

However, both cannot be zero simultaneously unless the solution is trivial (i.e., $A = B = 0$). Therefore, we check these conditions individually

- From $2B\sin(\beta\pi) = 0$, nontrivial solutions require $\sin(\beta\pi) = 0 \Rightarrow \beta = n$, $n \in \mathbb{Z}^+$. - Then $\cos(\beta\pi) = \cos(n\pi) = (-1)^n \neq 0$, so the derivative condition becomes $2B\beta(-1)^n = 0 \Rightarrow B = 0$, implying only cosine terms remain.

Alternatively, if $B \neq 0$, then we must choose $\cos(\beta\pi) = 0 \Rightarrow \beta = n + \frac{1}{2}$, which contradicts $\sin(\beta\pi) = 0$.

Therefore, valid nontrivial solutions occur when $\beta = n \in \mathbb{Z}^+$, leading to

$$\mu_n = n^2, \quad n = 1, 2, 3, \ldots$$

and the corresponding eigenfunctions

$$\eta_n(x) = \cos(nx), \quad \eta_n(x) = \sin(nx).$$

## Case 2: $\mu = 0$

The differential equation becomes

$$\frac{d^2\eta}{dx^2} = 0,$$

with general solution

$$\eta(x) = C_1 x + C_2.$$

Applying the periodic conditions

$$\eta(-\pi) = -C_1\pi + C_2, \quad \eta(\pi) = C_1\pi + C_2.$$

Setting $\eta(-\pi) = \eta(\pi)$ gives:

$$-2C_1\pi = 0 \Rightarrow C_1 = 0.$$

Then $\eta(x) = C_2$ is constant. Since $\eta'(x) = 0$, the derivative condition is also satisfied. Thus, $\mu = 0$ is an eigenvalue with constant eigenfunction

$$\eta(x) = 1.$$

## Case 3: $\mu < 0$

Let $\mu = -\beta^2$, where $\beta > 0$. The differential equation becomes

$$\frac{d^2\eta}{dx^2} - \beta^2\eta = 0,$$

with general solution

$$\eta(x) = Ae^{\beta x} + Be^{-\beta x}.$$

Applying periodic conditions

$$\eta(-\pi) = Ae^{-\beta\pi} + Be^{\beta\pi}, \quad \eta(\pi) = Ae^{\beta\pi} + Be^{-\beta\pi}.$$

Set $\eta(-\pi) = \eta(\pi)$

$$Ae^{-\beta\pi} + Be^{\beta\pi} = Ae^{\beta\pi} + Be^{-\beta\pi}.$$

This simplifies to

$$(A - B)(e^{\beta\pi} - e^{-\beta\pi}) = 0 \Rightarrow A = B.$$

Now, for the derivative condition

$$\eta'(-\pi) = A\beta e^{-\beta\pi} - B\beta e^{\beta\pi}, \quad \eta'(\pi) = A\beta e^{\beta\pi} - B\beta e^{-\beta\pi}.$$

Set $\eta'(-\pi) = \eta'(\pi)$, giving

$$(A + B)(e^{\beta\pi} - e^{-\beta\pi}) = 0 \Rightarrow A = -B.$$

Together, $A = B$ and $A = -B$ imply $A = B = 0$, giving only the trivial solution. Hence, there are no negative eigenvalues.

### Conclusion:

The eigenvalues are

$$\mu_n = n^2, \quad n = 0, 1, 2, \ldots$$

with corresponding eigenfunctions

$$\eta_0(x) = 1, \quad \eta_n(x) = \cos(nx), \quad \eta_n(x) = \sin(nx), \quad n = 1, 2, 3, \ldots$$

## 2.2 Observation of Eigenvalues and Eigenfunctions

For the periodic Sturm–Liouville problem considered, the set of eigenvalues is given by:

$$\mu_0 = 0, \quad \mu_n = n^2 \quad \text{for } n = 1, 2, 3, \ldots$$

The corresponding eigenfunctions associated with these eigenvalues are:

$$\eta_0(x) = 1, \quad \eta_n(x) = \cos(nx), \quad \eta_n(x) = \sin(nx), \quad n = 1, 2, 3, \ldots$$

**Definition:** Let $f(x)$ and $g(x)$ be real-valued functions that are integrable over an interval $I$. These functions are considered *orthogonal* on the interval $I$, with respect to a positive weight function $r(x) > 0$, if their inner product satisfies the condition

$$\int_I f(x)g(x)r(x)\,dx = 0.$$

$$\langle f, g\rangle = \int_I f(x)\, g(x)\, r(x)\, dx$$

evaluates to zero

$$\langle f, g\rangle = 0.$$

## 2.3 Some important Theorems

**Theorem 2.3.1.** *Let us examine a second-order linear differential equation of the Sturm–Liouville type defined over the closed interval $[a, b]$. The general form is expressed as*

$$\frac{d}{dx}\left(l(x)\frac{d\eta}{dx}\right) + m(x)\eta + \mu r(x)\eta = 0,$$

*where the coefficient functions $l(x), q(x)$, and the weight function $r(x)$ are continuous on $[a, b]$, with $l(x) > 0$ and $r(x) > 0$ for every $x \in [a, b]$.*

*Suppose $\eta_j$ and $\eta_k$ are eigenfunctions corresponding to distinct eigenvalues $\mu_j \neq \mu_k$, both continuously differentiable on $[a, b]$. Then, these eigenfunctions are orthogonal with respect to the weight $r(x)$, i.e.,*

$$\int_a^b \eta_j(x)\eta_k(x)r(x)\,dx = 0.$$

*Proof.* Let $\eta_j$ and $\eta_k$ satisfy the respective equations:

$$\frac{d}{dx}\left(l\frac{d\eta_j}{dx}\right) + (m + \mu_j n)\,\eta_j = 0,$$

$$\frac{d}{dx}\left(l\frac{d\eta_k}{dx}\right) + (m + \mu_k n)\,\eta_k = 0.$$

Multiply the first by $\eta_k$ and the second by $\eta_j$, then subtract:

$$\eta_k\frac{d}{dx}\left(l\frac{d\eta_j}{dx}\right) - \eta_j\frac{d}{dx}\left(l\frac{d\eta_k}{dx}\right) + (\mu_j - \mu_k)n\eta_j\eta_k = 0.$$

Using the product rule, rewrite as

$$(\mu_j - \mu_k)n\eta_j\eta_k = \frac{d}{dx}\left[l\left(\eta_k\frac{d\eta_j}{dx} - \eta_j\frac{d\eta_k}{dx}\right)\right].$$

Integrating over $[a, b]$:

$$(\mu_j - \mu_k)\int_a^b n\eta_j\eta_k\,dx = \left[l\left(\eta_k\frac{d\eta_j}{dx} - \eta_j\frac{d\eta_k}{dx}\right)\right]_a^b.$$

Assuming boundary conditions of the form

$$b_1\eta_j(b) + b_2\eta_j'(b) = 0, \quad b_1\eta_k(b) + b_2\eta_k'(b) = 0,$$

and similarly at $x = a$, it follows that the boundary term vanishes:

$$\left[l\left(\eta_k\frac{d\eta_j}{dx} - \eta_j\frac{d\eta_k}{dx}\right)\right]_a^b = 0.$$

Hence,

$$(\mu_j - \mu_k)\int_a^b n\eta_j\eta_k\,dx = 0.$$

Since $\mu_j \neq \mu_k$, it must be that

$$\int_a^b n\eta_j\eta_k\,dx = 0,$$

which establishes orthogonality. □

**Theorem 2.3.2.** *For the Sturm–Liouville problem on* $[a, b]$ *with*

$$\mathcal{L}[\eta] = -\frac{d}{dx}\left(l(x)\frac{d\eta}{dx}\right) + m(x)\eta,$$

*weight function* $n(x) > 0$*, and periodic boundary conditions*

$$\eta(a) = \eta(b), \quad \eta'(a) = \eta'(b),$$

*eigenfunctions* $\eta_n$ *and* $\eta_m$ *corresponding to distinct eigenvalues* $\mu_n \neq \mu_m$ *are orthogonal*

$$\int_a^b \eta_n(x)\,\eta_m(x)\,n(x)\,dx = 0.$$

*Proof.* Let $\eta_j$ and $\eta_k$ be eigenfunctions with eigenvalues $\mu_j$ and $\mu_k$, satisfying periodic boundary conditions

$$\eta_j(a) = \eta_j(b), \quad \eta_j'(a) = \eta_j'(b),$$

$$\eta_k(a) = \eta_k(b), \quad \eta_k'(a) = \eta_k'(b).$$

Using the previously derived identity

$$(\mu_j - \mu_k)\int_a^b n\eta_j\eta_k\,dx = \left[l(x)\left(\eta_k\frac{d\eta_j}{dx} - \eta_j\frac{d\eta_k}{dx}\right)\right]_a^b.$$

Due to periodicity of $l$, i.e., $l(a) = l(b)$, and the boundary conditions, the right-hand side is zero. Thus,

$$(\mu_j - \mu_k)\int_a^b n\eta_j\eta_k\,dx = 0,$$

which implies orthogonality for distinct eigenvalues. □

## 2.4 Singular Sturm–Liouville Systems

A Sturm–Liouville problem is termed *singular* if it possesses one or more of the following features

- The domain of the problem extends over a semi-infinite or fully infinite interval.
- The leading coefficient function $l(x)$ or the weight function $r(x)$ becomes zero at some point within the interval.
- One or more of the functions $l(x)$, $m(x)$, or $r(x)$ diverge at either endpoint of the interval.

When a singular Sturm–Liouville differential equation is paired with boundary conditions that are linear and homogeneous, the setup is referred to as a *singular Sturm–Liouville (SSL) system.* Such systems typically arise in situations where the differential equation exhibits singularities—like unbounded domains or coefficients that blow up or vanish—while the boundary conditions maintain the linear nature of the problem.

## 2.5 Example Related to Singular Sturm–Liouville Systems

**Example:** Consider the following singular Sturm–Liouville problem:

$$\frac{d}{dx}\left((1-x^2)\frac{d\eta}{dx}\right)+\mu\eta=0, \quad -1<x<1,$$

with the requirement that both $\eta(x)$ and $\eta'(x)$ remain bounded as $x \to \pm 1$.

**Solution:** In this case, the differential equation has the following coefficients:

$$l(x)=1-x^2, \quad r(x)=1.$$

Here, the function $l(x)$ approaches zero as $x \to \pm 1$, making the problem singular at both endpoints.

The functions that stay bounded over the entire interval are known as Legendre functions of the first kind, represented by $P_n(x)$, where $n = 0, 1, 2, \ldots$. These functions serve as the eigenfunctions of the associated problem, and their corresponding eigenvalues are given by:

$$\mu_n = n(n+1), \quad n = 0, 1, 2, \ldots$$

This singular Sturm–Liouville problem therefore yields an infinite sequence of real eigenvalues. The eigenfunctions $P_n(x)$ form an orthogonal set over $(-1, 1)$ with respect to the standard weight function $r(x) = 1$.

**Theorem 2.5.1.** *Let $\eta_1(x)$ and $\eta_2(x)$ be square-integrable eigenfunctions corresponding to distinct eigenvalues $\mu_1 \neq \mu_2$ of a (possibly singular) Sturm–Liouville problem:*

$$-\frac{d}{dx}\left(l(x)\frac{d\eta}{dx}\right)+m(x)\eta=\mu r(x)\eta,$$

*on an interval $[a, b]$, where $l(x) > 0$, $r(x) > 0$, and $m(x)$ are real-valued functions. Then the eigenfunctions $\eta_1$ and $\eta_2$ are orthogonal with respect to the weight function $r(x)$, i.e.,*

$$\int_a^b \eta_1(x)\eta_2(x)r(x)\,dx = 0.$$

*Proof.* Let the functions $\eta_j(x)$ and $\eta_k(x)$ be eigenfunctions associated with different eigenvalues $\mu_j \neq \mu_k$. They satisfy the differential equation

$$-\frac{d}{dx}\left(l(x)\frac{d\eta}{dx}\right) + m(x)\eta = \mu\, r(x)\eta,$$

where $l(x), m(x)$, and $r(x)$ are given coefficient functions. Multiply the equation for $\eta_j$ by $\eta_k$, and the equation for $\eta_k$ by $\eta_j$, then subtract the two resulting expressions. This yields

$$(\mu_j - \mu_k)\int_a^b r(x)\eta_j(x)\eta_k(x)\,dx = \left[l(x)\left(\eta_j(x)\eta_k'(x) - \eta_j'(x)\eta_k(x)\right)\right]_a^b.$$

If the boundary terms vanish—due to suitable conditions such as the finiteness of $\eta$ and $\eta'$ at singular endpoints or prescribed boundary conditions—then the right-hand side becomes zero, leading to:

$$(\mu_j - \mu_k)\int_a^b r(x)\eta_j(x)\eta_k(x)\,dx = 0.$$

Since $\mu_j \neq \mu_k$, we conclude

$$\int_a^b r(x)\eta_j(x)\eta_k(x)\,dx = 0.$$

□

# Chapter 3

# BVPs and its Applications

There are four types of BVPs.

### 3.0.1 $1^{st}$ BVP (The Dirichlet Problem)

Let $D \subset \mathbb{R}^2$ be a bounded domain with boundary $B = \partial D$. The goal is to find a function $\xi(x, y)$ such that

1. $\xi$ is harmonic in $D$, i.e.,
$$\Delta\xi = \frac{\partial^2 \xi}{\partial x^2} + \frac{\partial^2 \xi}{\partial y^2} = 0 \quad \text{in } D,$$
2. $\xi$ satisfies the boundary condition:
$$\xi(x, y) = f(s) \quad \text{for } (x, y) \in B,$$
where $f$ represents a continuous function defined along the boundary $B$.

This is a classical BVP for Laplace's equation.

### 3.0.2 $2^{nd}$ BVP (The Neumann problem)

This involves finding a function $\xi(x, y)$ such that it is harmonic in $D$ and satisfies
$$\frac{\partial \xi}{\partial n} = f(s) \quad \text{on } B,$$
with
$$\int_B f(s)\, ds = 0,$$
Here, the notation $\frac{\partial \xi}{\partial n}$ stands for the directional derivative of the function $\xi$ in the direction of the outward normal vector to the boundary $B$. This derivative quantifies the rate of change of $\xi$ normal to the boundary at each location.

### 3.0.3 $3^{rd}$ BVP (The Robin problem)

The problem consists of determining a function $\xi(x, y)$ that is harmonic within the domain $D$ and fulfills the boundary requirement

$$\frac{\partial \xi}{\partial n} + h(s)\,\xi = f(s) \quad \text{for } s \in B,$$

where the functions $f$ and $h$ are continuous and non-negative on the boundary $B$.

### 3.0.4 $4^{th}$ BVP (Mixed problem)

Find a function $\xi(x, y)$ that is harmonic within the domain $D$ and satisfies distinct types of boundary conditions on separate segments of the boundary $B$.

For instance, consider the following boundary conditions:

$$\begin{aligned} \xi &= g_1(s) \quad \text{for } s \in B_1, \\ \frac{\partial \xi}{\partial n} &= g_2(s) \quad \text{for } s \in B_2, \end{aligned}$$

where the boundary $B$ is split into two non-overlapping parts:

$$B = B_1 \cup B_2, \quad B_1 \cap B_2 = \emptyset.$$

## 3.1 Maximum and Minimum Principles

First we prove the maximum and minimum principles after this by using maximum and minimum principles, we prove the uniqueness and continuity theorems for the interior Dirichlet problem for two dimensional Laplace equation.

### 3.1.1 The Maximum Principle:

**Theorem 3.1.1** (Maximum Principle)**.** *Let $\xi(x, y)$ be a harmonic function defined on a bounded domain $D$, continuous on the closure $\overline{D} = D \cup B$, where $B$ is the boundary of $D$. Then $\xi$ attains its maximum value on the boundary $B$.*

*Proof.* Suppose, for the sake of contradiction, that $\xi$ achieves its maximum value $M_0$ at some interior point $P_0 = (x_0, y_0) \in D$, and not on the boundary $B$. Let $M = \max_{(x,y)\in B} \xi(x, y)$, so that $M_0 > M$.

Define the auxiliary function

$$v(x, y) = \xi(x, y) + \frac{M_0 - M}{4R^2} \left[(x - x_0)^2 + (y - y_0)^2\right],$$

where $R$ is chosen so that the domain $D$ is contained within the disk of radius $R$ centered at $(x_0, y_0)$.

Note that at $P_0$,

$$v(x_0, y_0) = \xi(x_0, y_0) = M_0,$$

while for any $(x, y) \in B$,

$$v(x, y) \leq M + \frac{M_0 - M}{4R^2} \cdot (2R)^2 = M + (M_0 - M) = M_0.$$

In fact, since the points on $B$ lie at least some distance away from $P_0$, this inequality is strict:

$$v(x, y) < M_0.$$

Thus, $v$ attains a strict maximum at the interior point $P_0$. However, calculating the Laplacian of $v$, we find:

$$v_{xx} + v_{yy} = \xi_{xx} + \xi_{yy} + \frac{M_0 - M}{R^2} = 0 + \frac{M_0 - M}{R^2} > 0,$$

because $\xi$ is harmonic ($\xi_{xx} + \xi_{yy} = 0$) and $M_0 > M$.

This contradicts the strong maximum principle, which states that a function with positive Laplacian cannot attain a local maximum in the interior unless constant. Hence, our initial assumption must be false, and $\xi$ attains its maximum only on the boundary $B$. □

### 3.1.2 Minimum Principle

**Theorem 3.1.2** (Minimum Principle)**.** *Let $\xi(x, y)$ be harmonic in the bounded domain $D$ and continuous on $\overline{D}$. Then $\xi$ attains its minimum value on the boundary $B$.*

*Proof.* Apply the Maximum Principle to the function $-\xi(x, y)$, which is also harmonic on $D$. The maximum of $-\xi$ on $\overline{D}$ occurs on $B$, implying the minimum of $\xi$ occurs on $B$. □

**Theorem 3.1.3** (Uniqueness Theorem)**.** *Whenever a solution to the Dirichlet problem exists, it is guaranteed to be unique.*

*Proof.* Assume that $\xi_1(x, y)$ and $\xi_2(x, y)$ are two functions solving the Dirichlet problem within the domain $D$ and adhering to the boundary conditions on $B$, such that

$$\nabla^2 \xi_1 = 0, \quad \nabla^2 \xi_2 = 0 \quad \text{in } D,$$

and

$$\xi_1 = f, \quad \xi_2 = f \quad \text{on } B.$$

Consider the difference $v(x, y) = \xi_1(x, y) - \xi_2(x, y)$. Since both $\xi_1$ and $\xi_2$ are harmonic, $v$ satisfies

$$\nabla^2 v = \nabla^2 \xi_1 - \nabla^2 \xi_2 = 0 \quad \text{in } D,$$

with zero boundary values

$$v = \xi_1 - \xi_2 = 0 \quad \text{on } B.$$

By the maximum and minimum principle for harmonic functions, the function $v$ must attain its maximum and minimum values on the boundary $B$. Given $v = 0$ on $B$, it follows that $v \equiv 0$ throughout $D$. Hence,

$$\xi_1(x, y) = \xi_2(x, y) \quad \text{for all } (x, y) \in D,$$

demonstrating the uniqueness of the solution. □

**Theorem 3.1.4** (Continuity Theorem)**.** *The solution to the Dirichlet problem changes smoothly and without sudden jumps when the boundary conditions are slightly varied*

*Proof.* Let $\xi_1$ and $\xi_2$ be solutions to Laplace's equation in $D$ corresponding to boundary data $f_1$ and $f_2$ respectively:

$$\nabla^2 \xi_1 = 0, \quad \xi_1 = f_1 \quad \text{on } B,$$

$$\nabla^2 \xi_2 = 0, \quad \xi_2 = f_2 \quad \text{on } B.$$

Define $v = \xi_1 - \xi_2$. Then $v$ is harmonic in $D$:

$$\nabla^2 v = 0 \quad \text{in } D,$$

and on the boundary satisfies

$$v = f_1 - f_2 \quad \text{on } B.$$

Suppose the boundary data are close, i.e., there exists $\varepsilon > 0$ such that

$$|f_1 - f_2| < \varepsilon \quad \text{on } B.$$

By the maximum principle, the maximum and minimum of $v$ occur on the boundary. Thus,

$$|v(x, y)| \leq \max_{(x,y)\in B} |f_1 - f_2| < \varepsilon,$$

for every $(x, y) \in D$.

Consequently,

$$|\xi_1(x, y) - \xi_2(x, y)| = |v(x, y)| < \varepsilon,$$

which confirms that the solution varies continuously with respect to the boundary data. □

## 3.2 Dirichlet Problem in a Circular Domain (Interior Problem)

We will now show that a solution to the Dirichlet problem exists within a circular domain Consider Laplace's equation in polar coordinates given by

$$\frac{\partial^2 \xi}{\partial r^2} + \frac{1}{r}\frac{\partial \xi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \xi}{\partial \theta^2} = 0, \quad 0 \leq r < a, \quad 0 \leq \theta \leq 2\pi, \tag{1}$$

with boundary condition

$$\xi(a, \theta) = f(\theta), \quad \theta \in [0, 2\pi]. \tag{2}$$

Our objective is to find a function $\xi(r, \theta)$ that solves Equation (1) within the disk of radius $a$ and matches the prescribed values on the circle $r = a$.

**Method of solution:** Suppose the solution is separable and can be written as

$$\xi(r, \theta) = R(r)\Theta(\theta), \quad R(r)\Theta(\theta) \neq 0.$$

Substituting into Equation (1) and dividing by $R\Theta$, we obtain

$$\frac{r^2 R'' + rR'}{R} = -\frac{\Theta''}{\Theta} = \mu,$$

where $\mu$ is a constant called the separation constant. This leads to the pair of ordinary differential equations:

$$r^2 R'' + rR' - \mu R = 0, \tag{3}$$

$$\Theta'' + \mu\Theta = 0. \tag{4}$$

The angular part is required to fulfill the periodic boundary conditions:

$$\Theta(0) = \Theta(2\pi), \quad \frac{d\Theta}{d\theta}(0) = \frac{d\Theta}{d\theta}(2\pi),$$

which guarantees the function is continuous and differentiable around the circle.

### Case 1: $\mu < 0$

This case yields exponential-type solutions for $\Theta$ that are not periodic, thus are discarded.

### Case 2: $\mu = 0$

Equation (4) becomes $\Theta'' = 0$, whose general solution is

$$\Theta(\theta) = A + B\theta.$$

For periodicity on $[0, 2\pi]$, it follows that $B = 0$, so $\Theta$ is constant. The radial equation (3) reduces to

$$r^2 R'' + rR' = 0,$$

whose general solution is

$$R(r) = C + D \ln r.$$

To avoid singularities at the origin ($r = 0$), we set $D = 0$. Thus, the solution is a constant:

$$\xi(r, \theta) = \text{constant}.$$

### Case 3: $\mu > 0$

Let $\mu = n^2$ for $n = 1, 2, 3, \ldots$ to satisfy periodicity. Then

$$\Theta(\theta) = A_n \cos n\theta + B_n \sin n\theta,$$

and the radial equation (3) is

$$r^2 R'' + rR' - n^2 R = 0,$$

an Euler equation with solutions

$$R(r) = C_n r^n + D_n r^{-n}.$$

To ensure boundedness at the origin, set $D_n = 0$. Hence,

$$R(r) = C_n r^n.$$

The general solution is thus represented as

$$\xi(r, \theta) = \frac{a_0}{2} + \sum_{n=1}^{\infty} r^n \left( a_n \cos n\theta + b_n \sin n\theta \right),$$

where $a_0, a_n, b_n$ are constants to be determined by the boundary condition.

Applying the boundary condition $\xi(a,\theta) = f(\theta)$, we get

$$f(\theta) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos n\theta + b_n \sin n\theta.$$

The coefficients are given by the Fourier formulas:

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(\theta)\cos n\theta \, d\theta, \quad n = 0, 1, 2, \ldots,$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(\theta)\sin n\theta \, d\theta, \quad n = 1, 2, 3, \ldots.$$

Therefore, the solution inside the disk is

$$\xi(r,\theta) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(\frac{r}{a}\right)^n \left(a_n \cos n\theta + b_n \sin n\theta\right).$$

### Convergence of the series

Since $f(\theta)$ is assumed integrable and bounded, the Fourier coefficients $a_n$ and $b_n$ are bounded. Let $M > 0$ be such that

$$|a_n| < M, \quad |b_n| < M, \quad \text{for all } n.$$

Define the sequence of functions

$$\xi_n(\rho,\theta) = \rho^n \left(a_n \cos n\theta + b_n \sin n\theta\right),$$

where $\rho = \frac{r}{a}$, with $0 \le \rho < 1$. Then

$$|\xi_n(\rho,\theta)| \le 2M\rho^n.$$

Since the geometric series $\sum \rho^n$ converges for $\rho < 1$, the series for $\xi(r,\theta)$ converges uniformly on any closed disk inside the open unit disk $\rho < 1$. This ensures the solution is well-defined and continuous in the interior of the domain.

## 3.3 Dirichlet Problem for a Circular Annulus

We extend the Dirichlet problem from a disk to an annular region defined by

$$\frac{\partial^2 \xi}{\partial r^2} + \frac{1}{r}\frac{\partial \xi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \xi}{\partial \theta^2} = 0, \quad r_2 < r < r_1, \tag{1}$$

subject to the boundary conditions

$$\xi(r_1, \theta) = f(\theta), \quad \xi(r_2, \theta) = g(\theta),$$

where $f(\theta)$ and $g(\theta)$ are $2\pi$-periodic functions in $\theta$.

**Solution:**

From the Dirichlet problem on a disk, the general form of the solution when the separation constant $\lambda = 0$ is

$$\xi(r, \theta) = (A + B \log r)(C\theta + D).$$

Since the solution must be periodic in $\theta$, it follows that $C = 0$. Thus,

$$\xi(r, \theta) = AD + BD \log r.$$

Defining constants $a_0 = 2AD$ and $b_0 = 2BD$, this becomes

$$\xi(r, \theta) = \frac{a_0}{2} + \frac{b_0}{2} \log r.$$

For $\lambda > 0$, with $\sqrt{\lambda} = n$ for $n = 1, 2, 3, \ldots$, the solution takes the form

$$\xi(r, \theta) = \left(Cr^n + Dr^{-n}\right)\left(A \cos n\theta + B \sin n\theta\right).$$

Therefore, the general solution in the annulus can be expressed as

$$\xi(r, \theta) = \frac{1}{2}(a_0 + b_0 \log r) + \sum_{n=1}^{\infty} \left[(a_n r^n + b_n r^{-n}) \cos n\theta + (c_n r^n + d_n r^{-n}) \sin n\theta\right],$$

where $a_n, b_n, c_n, d_n$ are constants to be determined.

**Determination of coefficients:**

Applying the boundary conditions yields the following system:

$$a_0 + b_0 \log r_1 = \frac{1}{\pi} \int_0^{2\pi} f(\tau)\, d\tau,$$

$$a_n r_1^n + b_n r_1^{-n} = \frac{1}{\pi} \int_0^{2\pi} f(\tau) \cos n\tau\, d\tau,$$

$$c_n r_1^n + d_n r_1^{-n} = \frac{1}{\pi} \int_0^{2\pi} f(\tau) \sin n\tau\, d\tau,$$

and similarly on the inner boundary,

$$a_0 + b_0 \log r_2 = \frac{1}{\pi} \int_0^{2\pi} g(\tau)\, d\tau,$$

$$a_n r_2^n + b_n r_2^{-n} = \frac{1}{\pi} \int_0^{2\pi} g(\tau) \cos n\tau\, d\tau,$$

$$c_n r_2^n + d_n r_2^{-n} = \frac{1}{\pi} \int_0^{2\pi} g(\tau) \sin n\tau\, d\tau,$$

for $n = 1, 2, 3, \ldots$.

This system can be solved for the coefficients $a_0, b_0, a_n, b_n, c_n, d_n$, fully determining the solution $\xi(r, \theta)$ inside the annulus.

## 3.4 Dirichlet Problem for a Rectangle

Consider the Laplace equation in the rectangular domain

$$\nabla^2 \xi = \xi_{xx} + \xi_{yy} = 0, \quad 0 < x < a, \quad 0 < y < b,$$

with boundary conditions

$$\xi(x, 0) = f(x), \quad \xi(x, b) = 0, \quad 0 \leq x \leq a,$$

$$\xi(0, y) = 0, \quad \xi(a, y) = 0, \quad 0 \leq y \leq b.$$

**Solution:**Assume a separable solution

$$\xi(x, y) = G(x)H(y).$$

Substituting into the Laplace equation, we get

$$G''(x)H(y) + G(x)H''(y) = 0.$$

Dividing both sides by $G(x)H(y) \neq 0$:

$$\frac{G''(x)}{G(x)} + \frac{H''(y)}{H(y)} = 0.$$

Set the separation constant $\mu$ as

$$\frac{G''(x)}{G(x)} = -\frac{H''(y)}{H(y)} = \mu.$$

This yields two ODEs

$$G''(x) - \mu G(x) = 0, \quad H''(y) + \mu H(y) = 0.$$

## Case (1): $\mu > 0$

Let $\mu = b^2$. Then

$$G''(x) - b^2 G(x) = 0$$

General solution:

$$G(x) = C_1 e^{bx} + C_2 e^{-bx}$$

Boundary conditions:

$$G(0) = 0 \Rightarrow C_1 = -C_2$$

$$G(a) = 0 \Rightarrow C_1(e^{ba} - e^{-ba}) = 0 \Rightarrow C_1 = 0 \Rightarrow G(x) = 0$$

$\Rightarrow \xi(x, y) = 0$

## Case (2): $\mu = 0$

$$G'' = 0 \Rightarrow G = Ax + B$$

Boundary conditions:

$$G(0) = 0 \Rightarrow B = 0, \quad G(a) = 0 \Rightarrow A = 0 \Rightarrow G(x) = 0 \Rightarrow \xi(x, y) = 0$$

## Case (3): $\mu < 0$

Let $\mu = -b^2$

$$G'' + b^2 G = 0 \Rightarrow G(x) = A\cos(bx) + B\sin(bx)$$

Boundary conditions:

$$G(0) = 0 \Rightarrow A = 0, \quad G(a) = B\sin(ba) = 0 \Rightarrow ba = n\pi \Rightarrow b = \frac{n\pi}{a}$$

$$\mu_n = -\left(\frac{n\pi}{a}\right)^2, \quad G_n(x) = B_n \sin\left(\frac{n\pi x}{a}\right)$$

Solve for $H(y)$:

$$H'' - b^2 H = 0 \Rightarrow H(y) = C_1 e^{by} + C_2 e^{-by}$$

$$H(a) = 0 \Rightarrow C_2 = -C_1 e^{2ba} \Rightarrow H(y) = C_1(e^{by} - e^{2ba-by})$$

Redefining:

$$H_n(y) = C_n\left(e^{\frac{n\pi}{a}y} - e^{\frac{n\pi}{a}(2a-y)}\right)$$

Then,

$$\xi(x, y) = \sum_{n=1}^{\infty} G_n(x) H_n(y)$$

$$= \sum_{n=1}^{\infty} B_n C_n \sin\left(\frac{n\pi x}{a}\right)\left(e^{\frac{n\pi}{a}y} - e^{\frac{n\pi}{a}(2a-y)}\right)$$

Let $A_n = B_n C_n$:

$$\xi(x, y) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{a}\right)\left(e^{\frac{n\pi}{a}y} - e^{\frac{n\pi}{a}(2a-y)}\right)$$

## Determining Coefficients $A_n$

Given $\xi(x, 0) = f(x)$:

$$f(x) = \sum_{n=1}^{\infty} A_n\left(1 - e^{2n\pi}\right)\sin\left(\frac{n\pi x}{a}\right)$$

Using orthogonality:

$$A_n(1 - e^{2n\pi}) = \frac{2}{a}\int_0^a f(x)\sin\left(\frac{n\pi x}{a}\right)dx$$

$$\Rightarrow A_n = \frac{2}{a(1 - e^{2n\pi})}\int_0^a f(x)\sin\left(\frac{n\pi x}{a}\right)dx$$

## Final Solution

$$\xi(x,y) = \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi x}{a}\right)\left[e^{\frac{n\pi}{a}y} - e^{\frac{n\pi}{a}(2a-y)}\right]$$

where

$$a_n = \frac{2}{a\left[1-e^{2n\pi}\right]}\int_0^a f(x)\sin\left(\frac{n\pi x}{a}\right)dx.$$

## 3.5 Neumann Problem for a Circle

Consider the BVP

$$\nabla^2\xi = \xi_{rr} + \frac{1}{r}\xi_r + \frac{1}{r^2}\xi_{\theta\theta} = 0, \quad r < a \quad \text{(interior domain)}.$$

The boundary condition is given by the Neumann condition

$$\left.\frac{\partial\xi}{\partial n}\right|_{r=a} = \left.\frac{\partial\xi}{\partial r}\right|_{r=a} = f(\theta),$$

where the compatibility condition

$$\int_0^{2\pi} f(\theta)\,d\theta = 0$$

must be satisfied to ensure solvability.

**Solution:** We look for a solution in the form of a Fourier series,

$$\xi(r,\theta) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(\frac{r}{a}\right)^n\left(a_n\cos n\theta + b_n\sin n\theta\right).$$

Applying the boundary condition at $r = a$,

$$\left.\frac{\partial\xi}{\partial r}\right|_{r=a} = \sum_{n=1}^{\infty}\frac{n}{a}\left(a_n\cos n\theta + b_n\sin n\theta\right) = f(\theta).$$

This implies that $f(\theta)$ admits the Fourier expansion

$$f(\theta) = \sum_{n=1}^{\infty}\frac{n}{a}\left(a_n\cos n\theta + b_n\sin n\theta\right).$$

By orthogonality of sine and cosine functions, the coefficients are given by

$$\frac{na_n}{a} = \frac{1}{\pi}\int_0^{2\pi} f(\theta)\cos n\theta\,d\theta, \quad \frac{nb_n}{a} = \frac{1}{\pi}\int_0^{2\pi} f(\theta)\sin n\theta\,d\theta.$$

Solving for $a_n$ and $b_n$, we get

$$a_n = \frac{a}{n\pi}\int_0^{2\pi} f(\theta)\cos n\theta\,d\theta, \quad b_n = \frac{a}{n\pi}\int_0^{2\pi} f(\theta)\sin n\theta\,d\theta.$$

Note that the constant $a_0$ remains undetermined, reflecting the fact that the Neumann problem determines the solution only up to an additive constant.

Substituting back into the series,

$$\xi(r,\theta) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(\frac{r}{a}\right)^n \left[\frac{a}{n\pi}\int_0^{2\pi} f(\tau)\cos n\tau\, d\tau \cos n\theta + \frac{a}{n\pi}\int_0^{2\pi} f(\tau)\sin n\tau\, d\tau \sin n\theta\right].$$

Rearranging the integrals and sums,

$$\xi(r,\theta) = \frac{a_0}{2} + \frac{a}{\pi}\int_0^{2\pi} f(\tau) \sum_{n=1}^{\infty} \frac{1}{n}\left(\frac{r}{a}\right)^n \cos\big(n(\theta-\tau)\big)\, d\tau.$$

Setting $\rho = \frac{r}{a}$, we use the identity

$$\sum_{n=1}^{\infty} \frac{\rho^n}{n}\cos n\phi = -\frac{1}{2}\log\left(1 - 2\rho\cos\phi + \rho^2\right),$$

to write the solution as

$$\xi(r,\theta) = \frac{a_0}{2} - \frac{a}{2\pi}\int_0^{2\pi} f(\tau)\log\left(a^2 + r^2 - 2ar\cos(\theta-\tau)\right) d\tau.$$

**The Neumann problem for exterior problem of circle is:** Consider the BVP

$$\nabla^2\xi = \xi_{rr} + \frac{1}{r}\xi_r + \frac{1}{r^2}\xi_{\theta\theta} = 0, \quad r > a \quad \text{(Exterior domain)}.$$

The boundary condition is given by the Neumann condition

$$\left.\frac{\partial\xi}{\partial n}\right|_{r=a} = \left.\frac{\partial\xi}{\partial r}\right|_{r=a} = f(\theta),$$

where the compatibility condition

$$\int_0^{2\pi} f(\theta)\, d\theta = 0$$

has solution

$$\xi(r,\theta) = \frac{a_0}{2} + \frac{a}{2\pi}\int_0^{2\pi} f(\tau)\log\left(a^2 + r^2 - 2ar\cos(\theta-\tau)\right) d\tau.$$

# Chapter 4

# Conclusion and Future Scopes

## 4.1 Conclusion

This dissertation presents a comprehensive analysis of SLtheory and its significant applications in solving BVPs. Starting with the mathematical foundation of eigenvalue problems, we developed the SLframework and examined both regular and singular forms. The orthogonality of eigenfunctions and the associated spectral properties were discussed in detail, with illustrative examples demonstrating their utility in classical problems.

In the context of BVPs, the study explored various types—including Dirichlet, Neumann, Robin, and mixed conditions—and applied these to geometries such as circles, rectangles, and annuli. Each formulation was solved analytically using separation of variables and Fourier series, leading to closed-form solutions where applicable. The maximum and minimum principles were used to establish uniqueness and continuity, reinforcing the theoretical robustness of the methods applied.

Overall, the work highlights how mathematical theory and physical application can be seamlessly integrated through analytical techniques rooted in SLproblems.

## 4.2 Future Scope

Although this thesis focused primarily on analytical methods, several avenues remain open for further exploration:

- Extending the analysis to nonhomogeneous and nonlinear boundary conditions.
- Studying numerical methods, such as finite difference or finite element approaches, for approximating solutions where analytical forms are difficult.

- Investigating SLproblems in higher dimensions and irregular domains.
- Exploring applications in quantum mechanics, wave propagation, and heat diffusion in more complex systems.

Such advancements would enhance the applicability of SLtheory to modern scientific and engineering problems, providing a bridge between classical analysis and computational techniques.

# Bibliography

[1] T. Myint-U and L. Debnath. *Linear partial differential equations for scientists and engineers*. Springer Science & Business Media, 2007.